
Non-spectrality of generators of some classical analytic semigroups

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Communicated by Prof. A.C. Zaanen at the meeting of October 30, 1989**ABSTRACT**

A simple proof is given of the facts that the infinitesimal generator of the heat semigroup and the Poisson semigroup are scalar operators in $L^p(\mathbb{R}^n)$, $1 < p < \infty$, $n \geq 1$, if and only if, $p = 2$. We exploit the fact that these semigroups consist of Fourier p -multiplier operators in the right-half plane which cannot be extended to a group of p -multiplier operators at the boundary $\operatorname{Re}(z) = 0$ if $p \neq 2$.

1. INTRODUCTION

Spectral operators of scalar-type (briefly, scalar operators), introduced by N. Dunford, are a generalization to Banach spaces of normal operators in Hilbert space. An appealing aspect of this class of operators is the extensive functional calculus that they admit. In the monograph [2] it is shown that a large class of operators in L^2 -spaces are spectral operators. Many such operators have natural analogues in the L^p -setting where they remain formally symmetric in the sense that the adjoint operator (acting in L^q where $p^{-1} + q^{-1} = 1$) is the “same operator” as the one acting in L^p . A typical example is the Laplace operator $\sum_{j=1}^n \partial^2 / \partial x_j^2$ in $L^p(\mathbb{R}^n)$, $1 < p < \infty$. By analogy with the L^2 -case it may be expected that such operators are scalar operators. Unfortunately, this is rarely the case. In general, it is usually necessary for the operator to satisfy additional criteria in order to be scalar.

In the article [8] a criterion was given characterizing generators of certain analytic semigroups as (unbounded) scalar operators with positive spectrum: a

necessary condition is that the semigroup be uniformly bounded in the right-half plane $\mathcal{H}^+ = \{z \in \mathbb{C}; \operatorname{Re}(z) > 0\}$. Such a (simple) necessary condition can sometimes be effectively used to show that generators of certain analytic semigroups in \mathcal{H}^+ which can be extended to the boundary of \mathcal{H}^+ (i.e. $z \in \mathbb{C}$ such that $\operatorname{Re}(z) = 0$) are not scalar operators. For example, if $\{T_z; z \in \mathcal{H}^+\}$ is the Riemann-Liouville semigroup in $L^p(0, 1)$, $1 < p < \infty$ (e.g. [5; Section 23.16]), then it is known that $\{T_z\}$ has an extension to a “boundary group” at $\operatorname{Re}(z) = 0$, [7]. In this case, it can be explicitly calculated that

$$\|T_{it}\|_{L^p} \geq \|T_{it}\|_{L^2} = e^{|t|\pi/2}, \quad t \in \mathbb{R},$$

[5; p. 665]. Accordingly, $\{T_z; \operatorname{Re}(z) \geq 0\}$ is not uniformly bounded and so the generator A of $\{T_z\}$ is not a scalar operator (this can also be deduced from the fact that $\sigma(A) = \emptyset$, [5; p. 664]). Unfortunately, such an example is rather special. It is usually difficult to compute $\|T_z\|$ precisely. Rather, one only has estimates of the form $\|T_z\| \leq M_z$, $z \in \mathcal{H}^+$. Even though $\sup\{M_z; z \in \mathcal{H}^+\}$ may be infinite it cannot be concluded directly that $\{T_z; \operatorname{Re}(z) \geq 0\}$ is unbounded. To deduce the desired contradiction it is often necessary to argue slightly differently.

In this note we wish to show that the “unboundedness criterion” of $\{T_z\}$ can be used to show that two classical infinitesimal generators, namely those of the heat semigroup and the Poisson semigroup in $L^p(\mathbb{R}^n)$, $1 < p < \infty$ with $p \neq 2$, though formally symmetric, are not scalar operators. The idea is a simple one: such semigroups consist of Fourier L^p -multiplier operators and hence, if they were uniformly bounded in \mathcal{H}^+ (a consequence of their generator being a scalar operator), then it would follow that the “boundary group” corresponding to $\operatorname{Re}(z) = 0$ also consists of Fourier L^p -multiplier operators. It will be shown that this is not the case.

2. THE HEAT SEMIGROUP

Let $1 < p < \infty$, n be a positive integer and $\Delta_{p,n} = \sum_{j=1}^n \partial^2 / \partial x_j^2$ denote the Laplace operator in $L^p(\mathbb{R}^n)$ with domain

$$\mathcal{D}(\Delta_{p,n}) = \{f \in L^p(\mathbb{R}^n); \Delta_{p,n}f \text{ exists in the weak sense and is in } L^p(\mathbb{R}^n)\}.$$

It is known that $\mathcal{D}(\Delta_{p,n})$ coincides with the Sobolev space $W^{2,p}(\mathbb{R}^n)$ and, if $1 < p \leq 2$, then also

$$\mathcal{D}(\Delta_{p,n}) = \{f \in L^p(\mathbb{R}^n); -|\xi|^2 \hat{f}(\xi) = \hat{g}(\xi) \text{ for some } g \in L^p(\mathbb{R}^n)\},$$

where $|\cdot|$ denotes the usual Euclidean norm in \mathbb{R}^n and $\hat{\cdot}$ denotes the Fourier transform. In this case $\Delta_{p,n}f = g$, for each $f \in \mathcal{D}(\Delta_{p,n})$, where g is the (unique) element of $L^p(\mathbb{R}^n)$ satisfying $\hat{g}(\xi) = -|\xi|^2 \hat{f}(\xi)$. It is known that $\Delta_{p,n}$ is a closed, densely defined operator (with spectrum $\sigma(\Delta_{p,n}) = (-\infty, 0]$) which generates an analytic semigroup $\{T_z; \operatorname{Re}(z) > 0\}$ in $L^p(\mathbb{R}^n)$ via convolution with the heat kernel

$$\psi_z(u) = (4\pi z)^{-n/2} \exp(-|u|^2/4z), \quad u \in \mathbb{R}^n.$$

That is, for a.e. $x \in \mathbb{R}^n$,

$$(T_z f)(x) = \int_{\mathbb{R}^n} f(u-x) \psi_z(u) du, \quad f \in L^p(\mathbb{R}^n).$$

Since $\psi_z \in L^1(\mathbb{R}^n)$, for $\operatorname{Re}(z) > 0$, the operator T_z is a Fourier L^p -multiplier operator: it corresponds to the p -multiplier

$$\hat{\psi}_z: w \mapsto \exp(-z|w|^2), \quad w \in \mathbb{R}^n.$$

In particular, if $\|\cdot\|_{p,n}$ denotes the p -multiplier norm (for the group \mathbb{R}^n) it follows that

$$\|T_z\|_{p,n} = \|\hat{\psi}_z\|_{p,n} \leq \|\psi_z\|_1 \leq [\cos(\arg(z))]^{-n/2},$$

for $|\arg(z)| < \frac{1}{2}\pi$, where $\|T_z\|_{p,n}$ denotes the operator norm of T_z considered in $L^p(\mathbb{R}^n)$.

Suppose that $\Delta_{p,n}$ is a scalar operator; for the definition and properties of such operators we refer to [2]. Let \mathcal{B} denote the Borel subsets of the complex plane \mathbb{C} and, for a Banach space X , let $B(X)$ denote the space of all bounded linear operators of X into itself. Then there exists a spectral measure $P: \mathcal{B} \rightarrow B(L^p(\mathbb{R}^n))$, with support $\sigma(\Delta_{p,n}) = (-\infty, 0]$, such that

$$\Delta_{p,n} = \int_{-\infty}^0 \lambda dP(\lambda).$$

Since $\lambda \mapsto \exp(z\lambda)$, $\lambda \leq 0$, is a bounded measurable function whenever $\operatorname{Re}(z) \geq 0$, it follows from the functional calculus for scalar operators that

$$S_z = \int_{-\infty}^0 \exp(z\lambda) dP(\lambda)$$

is an element of $B(L^p(\mathbb{R}^n))$, for each $z \in \mathcal{H}^+$, and that

$$\|S_z\|_{p,n} \leq 4K \sup\{|\exp(z\lambda)|; \lambda \leq 0\} = 4K,$$

for each $z \in \mathcal{H}^+$, where

$$K = \sup\{\|P(E)\|_{p,n}; E \in \mathcal{B}\} < \infty.$$

From the theory of \mathcal{C}_0 -semigroups it is known that

$$\lim_{m \rightarrow \infty} (I - t m^{-1} \Delta_{p,n})^{-m} = T_t, \quad t > 0,$$

where the limit exists in the strong operator topology. Since

$$\sup\{(1 - t\lambda m^{-1})^{-m}; \lambda \leq 0\} \leq 1,$$

for every $m = 1, 2, \dots$, and $t > 0$, and

$$\lim_{m \rightarrow \infty} (1 - t\lambda m^{-1})^{-m} = \exp(-\lambda t), \quad \lambda \leq 0,$$

it follows from the Dominated Convergence Theorem for vector measures, the σ -additivity of P (with respect to the strong operator topology) and the functional calculus for scalar operators that

$$\lim_{m \rightarrow \infty} (I - tm^{-1} \Delta_{p,n})^{-m} = \int_{-\infty}^0 \exp(\lambda t) dP(\lambda) = S_t,$$

for every $t > 0$, where again the limit exists in the strong operator topology. Accordingly, $T_t = S_t$, for each $t > 0$. But, $\{T_z; \operatorname{Re}(z) > 0\}$ is an analytic semigroup and, for each $f \in L^p(\mathbb{R}^n)$, it can be shown that $z \mapsto S_z f$, $z \in \mathcal{H}^+$, is also analytic (using the Dominated Convergence Theorem for the $L^p(\mathbb{R}^n)$ -valued measure $E \mapsto P(E)f$, $E \in \mathcal{B}$). Accordingly, by analytic continuation

$$T_z = S_z, \quad z \in \mathcal{H}^+,$$

from which it follows that

$$\sup\{\|\hat{\psi}_z\|_{p,n} = \|T_z\|_{p,n} = \|S_z\|_{p,n}; z \in \mathcal{H}^+\} \leq 4K < \infty.$$

Fix $y \in \mathbb{R}$ and let $z(m) = m^{-1} - iy$, for $m = 1, 2, \dots$. Then the sequence of p -multipliers $\{\hat{\psi}_{z(m)}\}_{m=1}^\infty$ converges pointwise on \mathbb{R}^n to the bounded function

$$(1) \quad w \mapsto \exp(iy|w|^2), \quad w \in \mathbb{R}^n,$$

and satisfies

$$\sup\{\|\hat{\psi}_{z(m)}\|_{p,n}; m = 1, 2, \dots\} < \infty.$$

Accordingly, as shown in Lemma 1.3 of [14], (1) is a p -multiplier in \mathbb{R}^n (for every $y \in \mathbb{R}$). For $p \neq 2$ this is known not to be the case [6]. This contradiction establishes the following result.

PROPOSITION 1. *Let n be a positive integer. If $1 < p < \infty$ and $p \neq 2$, then $\Delta_{p,n}$ is not a scalar-type spectral operator in $L^p(\mathbb{R}^n)$.*

REMARKS. (1) An alternative proof of Proposition 1, based on a Fuglede type result for intertwining of unbounded spectral operators, is given in [1]. The proof given there is based on the fact that if $f \in \mathcal{C}^2(\mathbb{R}^n, \mathbb{R})$ and there exist real numbers $t_n \rightarrow \infty$ such that each function $\varrho_n(x) = e^{it_n f(x)}$, $x \in \mathbb{R}^n$, is a p -multiplier and $\{\varrho_n\}$ is uniformly bounded in the p -multiplier norm, then $f(x) = \alpha_0 + \sum_{j=1}^n \alpha_j x_j$, $x \in \mathbb{R}^n$, for suitable real numbers α_j , $0 \leq j \leq n$. The proof given here rests simply on the fact that (1) is not a p -multiplier if $p \neq 2$, [6].

(2) For the case $n = 1$ other proofs of Proposition 1 can be found in [10, 11]. However, the arguments given there seem to be specific to $n = 1$.

(3) Let n be a positive integer. It is shown in [4] that the point spectrum of $\Delta_{p,n}$ is empty if $1 < p \leq 2n/(n-1)$ and is equal to the entire interval $(-\infty, 0)$ if $p > 2n/(n-1)$. Since the adjoint of a scalar operator in a reflexive Banach space is also a scalar operator and scalar operators have empty residual spectrum ($\lambda \in \mathbb{C}$ belongs to the residual spectrum of an operator T iff $T - \lambda I$ is injective and its range is not dense) it follows immediately (for $n \geq 2$) that $\Delta_{p,n}$ is not a

scalar operator for $p \in [2 - 2/(n+1), 2 + 2/(n-1)]$. Unfortunately this observation gives no information for $p \in [2 - 2/(n+1), 2 + 2/(n-1)]$ or when $n = 1$.

3. THE POISSON SEMIGROUP

Let $1 < p < \infty$, n be a positive integer and c_n be the constant as in [13; p. 61]. For $\operatorname{Re}(z) > 0$ define

$$\varphi_z(u) = c_n z(z^2 + |u|^2)^{-(n+1)/2}, \quad u \in \mathbb{R}^n.$$

Then $\varphi_z \in L^1(\mathbb{R}^n)$, for every $z \in \mathcal{H}^+$, and convolution with the Poisson kernel $\{\varphi_z; \operatorname{Re}(z) > 0\}$ generates an analytic semigroup $\{U_z; z \in \mathcal{H}^+\}$. That is, for a.e. $x \in \mathbb{R}^n$,

$$(\varphi_z f)(x) = \int_{\mathbb{R}^n} f(u - x) \varphi_z(u) du, \quad f \in L^p(\mathbb{R}^n).$$

Since $\varphi_z \in L^1(\mathbb{R}^n)$ the operator U_z is a Fourier L^p -multiplier operator, for every $z \in \mathcal{H}^+$: it corresponds to the p -multiplier

$$\hat{\varphi}_z: w \mapsto \exp(-z|w|), \quad w \in \mathbb{R}^n.$$

Let $A_{p,n}$ denote the generator of the Poisson semigroup $\{U_t; t > 0\} \subseteq \mathcal{B}(L^p(\mathbb{R}^n))$, in which case $A_{p,n}$ is a closed, densely defined operator and $\sigma(A_{p,n}) = (-\infty, 0]$. From semigroup theory it is known that

$$\mathcal{D}(A_{p,n}) = \{f \in L^p(\mathbb{R}^n); \lim_{t \rightarrow 0+} t^{-1}(U_t f - f) \text{ exists in } L^p(\mathbb{R}^n)\}$$

and, for every $f \in \mathcal{D}(A_{p,n})$,

$$(2) \quad A_{p,n} f = \lim_{t \rightarrow 0+} t^{-1}(U_t f - f).$$

LEMMA 1. *Let n be a positive integer, $1 < p \leq 2$ and*

$$V = \{f \in L^p(\mathbb{R}^n); -|\xi| \hat{f}(\xi) \text{ for some } g \in L^p(\mathbb{R}^n)\}.$$

Then $V = \mathcal{D}(A_{p,n})$ and, for each $f \in \mathcal{D}(A_{p,n})$, we have $A_{p,n} f = g$ where $g \in L^p(\mathbb{R}^n)$ satisfies $\hat{g}(\xi) = -|\xi| \hat{f}(\xi)$.

PROOF. If $f \in \mathcal{D}(A_{p,n})$, then it follows from (2) and continuity of the Fourier transform map $\hat{\cdot}: L^p(\mathbb{R}^n) \rightarrow L^q(\mathbb{R}^n)$, where $p^{-1} + q^{-1} = 1$, that

$$(A_{p,n} f)^\wedge = \lim_{t \rightarrow 0+} t^{-1}([U_t f]^\wedge - \hat{f}) = \lim_{t \rightarrow 0+} t^{-1}(e^{-t|\cdot|} \hat{f} - \hat{f});$$

the limit exists in $L^q(\mathbb{R}^n)$. Since

$$\lim_{t \rightarrow 0+} t^{-1}(e^{-t|w|} - 1) = -|w|, \quad w \in \mathbb{R}^n,$$

it follows that $f \in V$ and $(A_{p,n} f)^\wedge(\xi) = -|\xi| \hat{f}(\xi)$.

To establish the reverse inclusion let \tilde{A} denote the operator in $L^p(\mathbb{R}^n)$ with domain $\mathcal{D}(\tilde{A}) = V$ defined by $\tilde{A}f = g$ where, given $f \in V$, $g \in L^p(\mathbb{R}^n)$ is the unique element such that $\hat{g}(\xi) = -|\xi| \hat{f}(\xi)$. It was just shown that $\mathcal{D}(A_{p,n}) \subseteq \mathcal{D}(\tilde{A})$

and the restriction of \tilde{A} to $\mathcal{D}(A_{p,n})$ is $A_{p,n}$. In particular, \tilde{A} is densely defined. Since $1 \in \mathcal{Q}(A_{p,n})$ and $A_{p,n}$ is closed and densely defined it follows that $I - A_{p,n}$ maps $\mathcal{D}(A_{p,n})$ bijectively onto $L^p(\mathbb{R}^n)$. So, if $f \in \mathcal{D}(\tilde{A})$ there exists $h \in \mathcal{D}(A_{p,n})$ such that $(I - \tilde{A})f = (I - A_{p,n})h$. But, $h \in \mathcal{D}(A_{p,n})$ implies that $A_{p,n}h = \tilde{A}h$ and so $(I - \tilde{A})f = (I - \tilde{A})h$. That is, $(I - \tilde{A})(f - h) = 0$. By definition of \tilde{A} in terms of the Fourier transform it follows that

$$\xi \mapsto (1 + |\xi|)(\hat{f}(\xi) - \hat{h}(\xi)),$$

defined for a.e. $\xi \in \mathbb{R}^n$, is the zero element of $L^q(\mathbb{R}^n)$ and hence, $\hat{f} = \hat{h}$ in $L^q(\mathbb{R}^n)$. That is, $f = h \in \mathcal{D}(A_{p,n})$. \square

PROPOSITION 2. *Let n be a positive integer. If $1 < p < \infty$ and $p \neq 2$, then $A_{p,n}$ is not a scalar-type spectral operator in $L^p(\mathbb{R}^n)$.*

PROOF. Assume $1 < p < 2$. By definition of positive integral powers of unbounded operators we have

$$\mathcal{D}(A_{p,n}^2) = \{f \in L^p(\mathbb{R}^n); A_{p,n}f \in \mathcal{D}(A_{p,n})\}.$$

It follows from Lemma 1 that

$$\mathcal{D}(A_{p,n}^2) = \{f \in L^p(\mathbb{R}^n); |\xi|^2 \hat{f}(\xi) = \hat{g}(\xi) \text{ for some } g \in L^p(\mathbb{R}^n)\}$$

and, for each $f \in \mathcal{D}(A_{p,n}^2)$, $A_{p,n}^2 f = g$ where $g \in L^p(\mathbb{R}^n)$ satisfies $\hat{g}(\xi) = |\xi|^2 \hat{f}(\xi)$. Recalling (for $1 < p \leq 2$) the definition of $A_{p,n}$ and $\mathcal{D}(A_{p,n})$ in terms of the Fourier transform it follows that $A_{p,n}^2 = -A_{p,n}$ with equality of domains. Since a polynomial function of a scalar operator is also a scalar operator, Proposition 1 implies that $A_{p,n}$ is not a scalar operator. The case $2 < p < \infty$ follows from the facts that $A_{p,n}$ is formally symmetric and the adjoint of a scalar operator in a reflexive Banach space is also a scalar operator. \square

It is (perhaps) also worthwhile to give a direct proof of Proposition 2 because it provides a nice illustration of applying stationary phase analysis (c.f. Lemma 2) to establish that certain bounded functions are not p -multipliers; see also [12], for example.

So, to establish Proposition 2 via this method suppose that $A_{p,n}$ is a scalar operator. Then there is a spectral measure $Q: \mathcal{B} \rightarrow B(L^p(\mathbb{R}^n))$ such that

$$A_{p,n} = \int_{-\infty}^0 \lambda dQ(\lambda).$$

Let $1 < p < 2$. Arguing along the lines of the proof of Proposition 1 it follows that the operators

$$W_z = \int_{-\infty}^0 \exp(z\lambda) dQ(\lambda), \quad z \in \mathcal{H}^+,$$

defined via the functional calculus for scalar operators are elements of $B(L^p(\mathbb{R}^n))$, that $z \mapsto W_z$ is analytic in \mathcal{H}^+ and that

$$\sup\{\|W_z\|_{p,n}; z \in \mathcal{H}^+\} < \infty.$$

Again arguing as in the proof of Proposition 1 it can be shown that W_t is precisely the Fourier L^p -multiplier operator U_t , for each $t > 0$, and hence (by analytic continuation) $W_z = U_z$, for every $z \in \mathcal{H}^+$. In particular,

$$\sup\{\|\hat{\phi}_z\|_{p,n} = \|U_z\|_{p,n} = \|W_z\|_{p,n}; z \in \mathcal{H}^+\} < \infty.$$

Fix $t \in \mathbb{R}$ and let $z(m) = m^{-1} - it$, for each $m = 1, 2, \dots$. Then the sequence of p -multipliers $\{\hat{\phi}_{z(m)}\}_{m=1}^\infty$ converges pointwise on \mathbb{R}^n to the bounded function

$$(3) \quad w \mapsto \exp(it|w|), \quad w \in \mathbb{R}^n,$$

and satisfies

$$\sup\{\|\hat{\phi}_{z(m)}\|_{p,n}; m = 1, 2, \dots\} < \infty.$$

It follows that (3) is a p -multiplier in \mathbb{R}^n (for every $t \in \mathbb{R}$) which contradicts the following fact (for $n \geq 2$), thereby completing the proof.

LEMMA 2. *Let $n \geq 2$ be an integer. If $t \in \mathbb{R} \setminus \{0\}$, then (3) is not a p -multiplier in \mathbb{R}^n for any $1 < p < \infty$ other than $p = 2$.*

The proof of Lemma 2 uses the following result from stationary phase analysis; see [9; p. 41].

LEMMA 3. *Let n be a positive integer, $P: \mathbb{R}^n \rightarrow \mathbb{R}$ be a C^∞ -function and u be a rapidly decreasing function such that \hat{u} has compact support and*

$$(4) \quad \text{supp}(\hat{u}) \cap \{x \in \mathbb{R}^n; \det(\partial^2 P / \partial x_j \partial x_k) = 0\}$$

is empty. For $t \in \mathbb{R}$ define

$$u_t(x) = (2\pi)^{-n/2} \int_{\mathbb{R}^n} \exp(i(\langle x, w \rangle - tP(w))) \hat{u}(w) dw, \quad x \in \mathbb{R}^n,$$

where $\langle x, w \rangle = \sum_{j=1}^n x_j w_j$. Then there exists a constant $c(u)$ such that

$$|u_t(x)| \leq c(u) |t|^{-n/2}, \quad x \in \mathbb{R}^n,$$

for all $|t| > 1$.

PROOF OF LEMMA 2. It suffices to consider $2 < p < \infty$. Let $R > 1$ and u be a function as in Lemma 3 such that $\|u\|_p > 0$ and $\text{supp}(\hat{u}) \subseteq \{x \in \mathbb{R}^n; |x| \geq R\}$. Since $p > 2$ it follows that

$$(5) \quad \|u_t\|_p^p \leq \|u_t\|_\infty^{p-2} \|u_t\|_2^2.$$

Choose a real-valued $\varphi \in C^\infty(\mathbb{R}^n)$ such that $\varphi = 0$ in some (small) ball around zero and $\varphi(x) = 1$, for all $|x| > R$. Let $P(x) = |x| \varphi(x)$, for each $x \in \mathbb{R}^n$, in which case P is C^∞ and (4) is satisfied provided $n \geq 2$. Now, for $|t| > 1$ Lemma 3 implies that $\|u_t\|_\infty \leq c(u) |t|^{-n/2}$ and so (5) implies that

$$\|u_t\|_p^p \leq c(u)^{p-2} \|u_t\|_2^2 |t|^{-n(p-2)/2}.$$

Since $\|u_t\|_2 = \|u\|_2$ (by Parseval's formula) it follows that

$$\|u_t\|_p \leq c(u, p) |t|^{n(p^{-1}-1/2)}, \quad |t| > 1,$$

where $c(u, p) = \|u\|_2^{1/p} c(u)^{(p-2)/p}$.

Now, for u and P as chosen above we have

$$u_t(x) = (2\pi)^{-n/2} \int_{\mathbb{R}^n} \exp(i(\langle x, w \rangle - t|w|)) \hat{u}(w) dw,$$

for each $x \in \mathbb{R}^n$. Noting that

$$\hat{u}_t(\xi) = e^{it|\xi|} \hat{u}(\xi), \quad \xi \in \mathbb{R}^n,$$

it can be argued as in the proof of Lemma 1.3 in [6] that if (3) is a p -multiplier for some $t \neq 0$, then it is a p -multiplier for all $t \in \mathbb{R}$ and the p -multiplier norms are uniformly bounded in t . So, there is $\kappa(u) > 0$ such that

$$\|u\|_p \leq \kappa(u) \|u_t\|_p, \quad t \in \mathbb{R}.$$

It follows that

$$\|u\|_p \leq \kappa(u) c(u, p) |t|^{n(p^{-1}-1/2)}, \quad |t| > 1,$$

which is a contradiction as $\|u\|_p > 0$ and $(p^{-1} - 1/2) < 0$. Accordingly, (3) is not a p -multiplier whenever $t \neq 0$. \square

REMARK (4). The condition (4) is never satisfied for the specified P in the case of $n = 1$ since

$$\{x \in \mathbb{R}; \det(d^2 P/dx^2) = 0\} = \mathbb{R}.$$

This is not surprising since, in \mathbb{R}^1 , the functions (3) are p -multipliers for every $1 < p < \infty$. This is seen from the identities

$$e^{it|w|} = e^{itw} \chi_{[0, \infty)}(w) + e^{-itw} \chi_{(-\infty, 0)}(w), \quad w \in \mathbb{R},$$

valid for every $t \in \mathbb{R}$. So, for $n = 1$, the “boundary group” for U_z at $Re(z) = 0$ is uniformly bounded in $L^p(\mathbb{R})$, $1 < p < \infty$. However, the first proof of Proposition 2 is still valid and so $A_{p,1}$ is not a scalar operator in $L^p(\mathbb{R})$ whenever $p \neq 2$.

Another proof of Proposition 2 is given in [1]; it is based (indirectly) on the fact [3] that the characteristic function of the unit ball in \mathbb{R}^n , $n \geq 2$, is not a p -multiplier ($p \neq 2$). However, this is a rather deep result. The proof given here rests on the more easily verifiable fact that (3) is not a p -multiplier ($p \neq 2, n \geq 2$).

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